2021
MATHEMATICS
[HONOURS]
Paper : V

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.
Candidates are required to give their answers in their own words as far as practicable.
Notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) If J be the Jacobian of the system u, v with regard to x, y and J' be the Jacobian of x, y with regard to u, v then prove that $J.J'=1$.

b) Using implicit function theorem, show that $xy\sin x+\cos y=0$. Determine $y=\phi(x)$ uniquely near $\left(0,\frac{\pi}{2}\right)$ and find $\frac{dy}{dx}$ at $\left(0,\frac{\pi}{2}\right)$.

c) Give an example of a function which is not integrable.

d) Prove that,

$$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x.$$

e) For any bounded function f on [a, b] prove that, $\underline{\int_{a}^{b}} f(x)dx \le \overline{\int_{a}^{b}} f(x)dx$.

f) Show that, $f(x,y)=y^2+x^2y+x^4$ has a minimum at (0, 0).

g) Examine whether fundamental theorem of integral calculus is applicable to evaluate the integral $\int_0^3 x[x]dx$.

h) Evaluate: $\int_0^\infty\left(\frac{\sin x}{x}\right)^2 dx$

i) Is the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}\ (-\infty<x<\infty)$ converges uniformly? Justify.

j) Find the radius of convergence of the power series $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}z^n$.

k) Explain: Every Cauchy sequence may not be convergent.

l) Give example to show that, if two sets are separated then their components may or may not be separated.

m) For any metric space (X, d) prove that, $(A\cap B)^* = (A)^* \cap (B)^*;\ A, B \subseteq X$.

n) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space (x, d). If $d(x_n, y_n) \to 0$ in $\mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence in X, then show that $\{y_n\}$ is also a Cauchy sequence in X.

o) Show that, $\sin \overline{Z}$ is not analytic in $\mathbb{C}$.

p) Prove that a bilinear transformation other than the identity map has atmost two fixed points.

2. Answer any **five** questions: 8×5=40

a) i) State and prove the first mean value theorem of integral calculus. Hence deduce fundamental theorem of integral calculus from it.

ii) If $f:[a,b]\to\mathbb{R}$ is continuous and $f(x)>0,\ \forall x\in[a,b]$ and $F(x)=\int_a^x f(t)dt,\ x\in[a,b]$. Show that, F is strictly increasing in $[a,b]$.

(1+3+2)+2

b) i) Let f be a function defined and bounded on $[a,b]$ such that f has infinite number of points of finite discontinuity in $[a,b]$ but the set of all these points of discontinuity of f in $[a,b]$ has a finite number of limiting points. Prove that, f is Riemann integrable over $[a,b]$.

ii) Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$. Let $v:[c,d]\to\mathbb{R}$ be differentiable on [c, d] and $v([c,d])\subseteq[a,b]$. If $G:[c,d]\to\mathbb{R}$ be defined by $G(x)=\int_a^{v(x)} f(t)dt, x\in[c,d]$, then show that, $G'(x)$ exists and $G'(x)=f[v(x)]v'(x)$. 5+3

c) i) Prove that, $\frac{-1}{2} < \int_0^1 \frac{x^3 \cos 5x}{2+x^2} dx < \frac{1}{2}$.

ii) Evaluate the limit,

$\operatorname{Lt}_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{n+n}\right]$, as an integral.

iii) Using 2nd Mean Value Theorem due to Bonnet, show that, $\exists\, \xi \in \left[0, \frac{\pi}{2}\right]$ such that, $\int_0^{\frac{\pi}{2}} e^{-x} \cos x\, dx = \sin \xi$. 3+3+2

d) i) If $\{f_n\}$ be a sequence of Riemann integrable functions on [a, b] such that $\{f_n\}$ converges uniformly to f on [a, b]. Then prove that, f is Riemann integrable on [a, b] and the sequence $\left\{\int_a^b f_n\right\}$ converges to $\int_a^b f$.

ii) Let $Q = \{x_1, x_2, ..., x_n, ...\}$ be the countable set of rational numbers in [0, 1] and let $f_n : [a, b] \to \mathbb{R}$ be defined by, $f_n(x) = \begin{cases} 1 & ;\ x = x_1, x_2, ...x_n \\ 0 & ;\ \text{elsewhere} \end{cases}$

State with reason whether,

I) the point-wise limit of $\{f_n\}$ is Riemann integrable on [0, 1].

II) $\{f_n\}$ converges uniformly on [0, 1]. 5+3

e) i) Prove the inequality $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ where $x \geq 0, y \geq 0, z \geq 0$, using the method of Lagrange's undetermined multipliers.

ii) If the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on [a, b], prove that the series $\sum_{n=1}^{\infty} g(x) f_n(x)$ is uniformly convergent on [a, b] where g is bounded on [a, b]. 4+4

f) i) For the function,

$$f(x,y)=\begin{cases} x^2\tan^{-1}\left(\dfrac{y}{x}\right)-y^2\tan^{-1}\left(\dfrac{x}{y}\right); & \text{when } x\neq 0,\ y\neq 0 \\ 0 \quad ; & \text{elsewhere}\end{cases}$$

Show that, $f_{xy}(0,0)\neq f_{yx}(0,0)$

ii) Show that the integral $\int_0^{\frac{\pi}{2}} \log\sin x\,dx$ is convergent and hence evaluate it.

3+5

g) i) Assuming the validity of differentiation under the sign of integration prove that,

$$\int_0^{\pi}\frac{\log(1+\beta\cos x)}{\cos x}dx=\pi\sin^{-1}\beta\,(|\beta|<1)$$

ii) Prove that $\displaystyle\int_0^{\pi}\frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}dx=\frac{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}{2\sqrt{2\pi}}$.

5+3

h) i) If a_n and b_n are Fourier co-efficients of f then prove that,

$$\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\leq\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx$$

ii) Show that the Fourier series of f, where $f(x)=x^2$ in $(-\pi,\pi]$ and $f(x+2\pi)=f(x)$,

$\forall x$; is $\displaystyle\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$ 4+4

3. Answer any **three** questions: 8×3=24

a) i) Let A denotes the set of all sequences of complex numbers. Define a function $d: A\times A\to\mathbb{R}$ by,

$$d(x,y)=\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\cdot\frac{|x_n-y_n|}{1+|x_n-y_n|}\right)$$

for any $x=\{x_n\},\ y=\{y_n\}\in A$.

Prove that d is a metric in A.

ii) Show that Q×Q is not open in $\mathbb{R}^2$ with respect to the usual metric. 5+3

b) i) Prove that, in any metric space every open sphere is a open set.

ii) A subset S of a metric space (x, d) is dense in X if and only if every non-empty open ball intersects S. 4+4

c) i) Show that $(C[-1,1],d)$ is incomplete with respect to the metric d, given by,

$$d(f,g)=\int_{-1}^{1}|f(x)-g(x)|dx;\forall f,g\in C[-1,1].$$

ii) Prove that a second countable metric space is separable. 4+4

d) i) Let (X, d) and (Y,d') be two metric spaces. Show that a function $f:(X,d)\to(Y,d')$ is continuous iff for all closed set G in (Y,d'), $f^{-1}(G)$ is closed in (X, d).

ii) Show that, $S=\{n+m\sqrt{2}:n,m\in\mathbb{Z}\}$ is dense in $\mathbb{R}$. 4+4

e) i) Prove that intersection of a finite number of open sets is open. Give an example to show that this result is not true for infinite numbers of open sets.

ii) If (X, d) has only a finite number of components, then all the components are open. 5+3

4. Answer any **two** questions: 8×2=16

a) i) Determine where the function $f(z)=z+\frac{1}{z}$ is conformal and where it is not.

ii) State and prove Cauchy-Hadamard theorem on the radius of convergence of a power series in the complex plane. 2+6

b) i) If $a\in\mathbb{R}$, then show that, $u(x,y)=e^{-2axy}\cos a(x^2-y^2)$, is harmonic in $\mathbb{R}^2$. Find all its harmonic conjugates v(x, y) in $\mathbb{R}^2$.

ii) If z_1 and z_2 are the roots of the quadratic equation $\alpha z^2+2\beta z+\gamma=0$, then prove that,

$$|z_1|+|z_2|=\frac{1}{|\alpha|}\left\{\left|-\beta+\sqrt{\alpha\gamma}\right|+\left|-\beta-\sqrt{\alpha\gamma}\right|\right\}.$$

4+4

c) i) Show that the bilinear transformation which carries the points z = i, 0, –i respectively into $\omega=0,-1,\infty$ maps the upper half plane $\operatorname{Im}(z)>0$ on $|\omega|<1$.

ii) Show that a necessary condition that the function $f = u + iv$ is differentiable at a point $z_0 = x_0 + iy_0$ is that the partial derivatives u_x, u_y, v_x, v_y exist and $u_x = v_y, u_y = -v_x$ at (x_o, y_o). 4+4

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